Data assimilation using spectral approximation of covariance in EnKF

Ivan Kasanický

September 11, 2015

Team and support

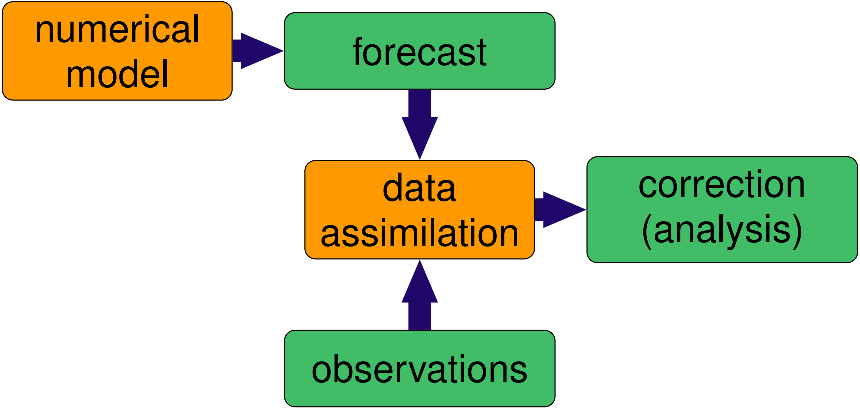
Joint work with Jan Mandel, Martin Vejmelka, Kryštof Eben and Pavel Juruš

Supported by Czech Science Foundation grant no. GA13-34856S and the US National Science Foundation under the grant DMS-1216481.

Content

1. Motivation, explanation
2. Data assimilation – mathematical formulation
3. Our contribution
4. Experiments results

Data assimilation cycle



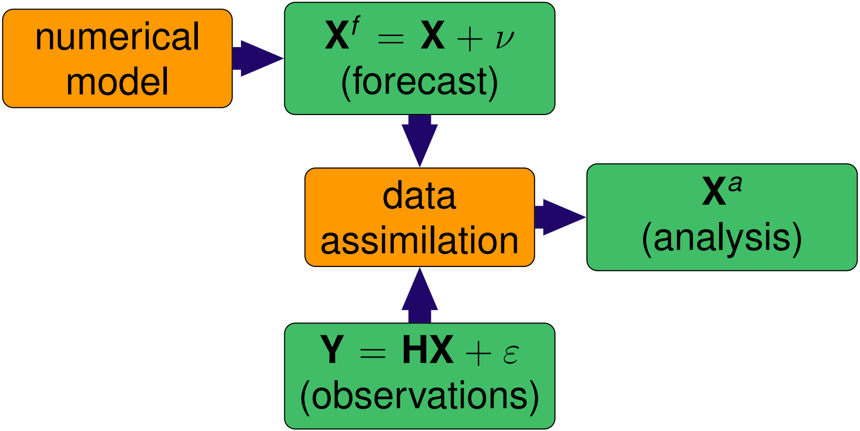
State-space model

- True (unobserved) state
  $\mathbf{X}_t \in \mathbb{R}^n$, $n \sim 10^{6-9}$
  $\mathcal{M} : \mathbf{X}_{t_1} \to \mathbf{X}_{t_2}$

- Noisy observations available
  $\mathbf{Y}_t = \mathbf{H}\mathbf{X}_t + \varepsilon$, $\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t)$
  $\mathbf{Y}_t \in \mathbb{R}^m$, $m \sim 10^{2-5}$

- **Goal:** create the best estimate of $\mathbf{X}$ based on forecast $\mathbf{X}_t^f$ and observations $\mathbf{D}_t$
  $\mathbf{X}^f = \mathbf{X} + \nu$, $\nu \sim \mathcal{N}(\mathbf{X}_t, \mathbf{Q}_t)$

Data assimilation cycle



- Cost function

$$\begin{aligned}\mathcal{J}(\mathbf{X}) &= (\mathbf{X}-\mathbf{X}^f)^\top \mathbf{B}^{-1}(\mathbf{X}-\mathbf{X}^f) \\ &\quad + (\mathbf{D}-\mathbf{H}\mathbf{X})^\top \mathbf{R}^{-1}(\mathbf{D}-\mathbf{H}\mathbf{X}) \\ &= \frac{1}{2}||\mathbf{X}-\mathbf{X}^f||_{\mathbf{B}} + \frac{1}{2}||\mathbf{D}-\mathbf{H}\mathbf{X}||_{\mathbf{R}}\end{aligned}$$

- Find $\mathbf{X}$, for which $\mathcal{J}(\mathbf{X})$ is minimal
- No time dependences!

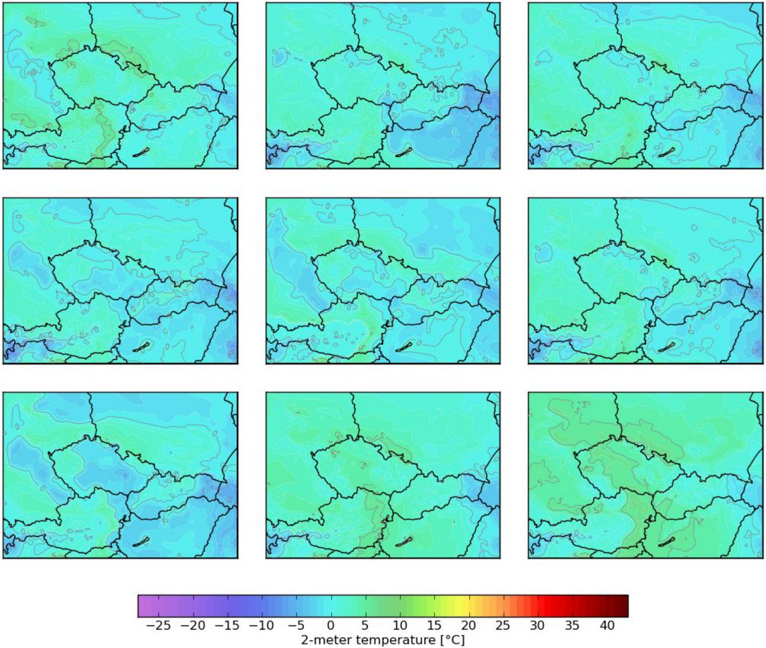
$$\mathbf{X}^a = \mathbf{X}^f + \underbrace{\mathbf{Q}\mathbf{H}^\top(\mathbf{H}\mathbf{Q}\mathbf{H}^\top + \mathbf{R})^{-1}}_{\text{Kalman gain}} \underbrace{(\mathbf{Y} - \mathbf{H}\mathbf{X}^f)}_{\text{Innovations}}$$

$$\mathbf{R} = cov(\mathbf{Y})$$

$$\mathbf{Q} = cov(\mathbf{X}^f) \qquad (10^9 \times 10^9)$$

- Best linear unbiased estimate
- If all distributions are Gaussian, analysis distribution remains Gaussian

2013-12-13_14:00:00



Ensemble Kalman filter

- Represent the distribution by an ensemble
  $\mathbf{X}_1^f, \ldots, \mathbf{X}_N^f$ (distribution of forecast)
  $\mathbf{X}_1^a, \ldots, \mathbf{X}_N^a$ (distribution of analysis
- Update formula

$$\mathbf{X}_i^a = \mathbf{X}_i^f + \mathbf{Q}_N \mathbf{H}^\top (\mathbf{H}\mathbf{Q}_N\mathbf{H}^\top + \mathbf{R})^{-1}(\mathbf{Y}_i - \mathbf{H}\mathbf{X}_i^f)$$

$$\mathbf{Q}_N = \frac{1}{N-1}\sum_{i=1}^{N}(\mathbf{X}_i^f - \bar{\mathbf{X}}^f)(\mathbf{X}_i^f - \bar{\mathbf{X}}^f)^\top$$

- Data must be perturbed!
  $\mathbf{Y}_i = \mathbf{Y} + \varepsilon$, $\varepsilon \sim \mathcal{N}(0, \mathbf{R})$

Singular value decomposition

If **W** is second order stationary field

$$cov(\mathbf{W}(y_1), \mathbf{W}(y_2)) = f(y_1 - y_2)$$

then

$$cov(\mathbf{W}) = \mathbf{F}^\top \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \mathbf{F}$$

eigenvectors of $cov(\mathbf{W})$ are **Fourier basis vectors**

Spectral diagonal EnKF

Transform the ensemble to spectral space

$$\mathbf{Z}_1^f = \mathbf{F}\mathbf{X}_1^f, \ldots, \mathbf{Z}_N^f = \mathbf{F}\mathbf{X}_N^f$$

$$cov(\mathbf{Z}^f) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} = \mathbf{F}cov(\mathbf{X})\mathbf{F}^\top$$

Spectral diagonal EnKF

$$\mathbf{D}_N = \mathbf{F}^\top \underbrace{\left(\sum_{i=1}^{N} \frac{(\mathbf{Z}_i^f - \bar{\mathbf{Z}}^f)(\mathbf{Z}_i^f - \bar{\mathbf{Z}}^f)^\top}{N-1} \circ \mathbf{I} \right)}_{\Lambda_N} \mathbf{F}$$

- Update formula

$$\mathbf{X}_i^a = \mathbf{X}_i^f + \mathbf{D}_N \mathbf{H}^\top (\mathbf{H}\mathbf{D}_N\mathbf{H}^\top + \mathbf{R})^{-1} (\mathbf{Y}_i - \mathbf{H}\mathbf{X}_i^f)$$

- Special case, $\mathbf{H} = \mathbf{I}$ and $\mathbf{R} = \mathbf{I}$

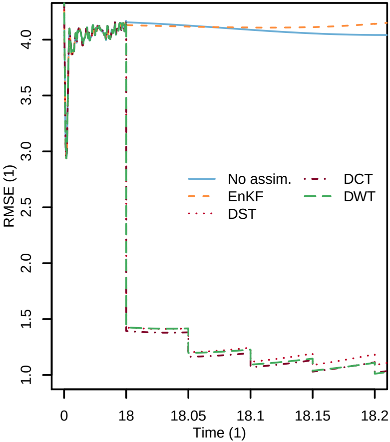
$$\mathbf{F}\mathbf{X}_i^a = \mathbf{F}\mathbf{X}_i^f + \Lambda(\Lambda + \mathbf{I})^{-1} \mathbf{F}(\mathbf{Y}_i - \mathbf{X}_i^f)$$

Optimality of spectral diagonal EnKF

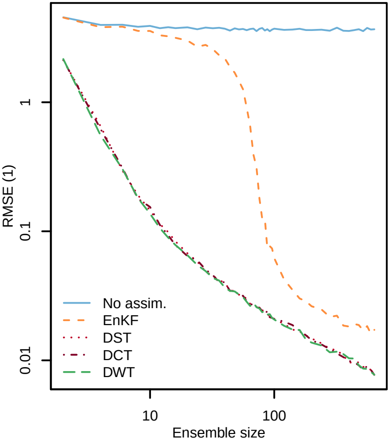
$$E\left[\|\mathbf{Q}-\mathbf{D}_N\|_{\mathrm{F}}^2\right] \leq E\left[\|\mathbf{Q}-\mathbf{C}_N\|_{\mathrm{F}}^2\right]$$

Frobenius norm of matrix:
$\|\mathbf{A}\|_{\mathrm{F}} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{i,j}|^2\right)^{1/2}$

Lorenz 96 model



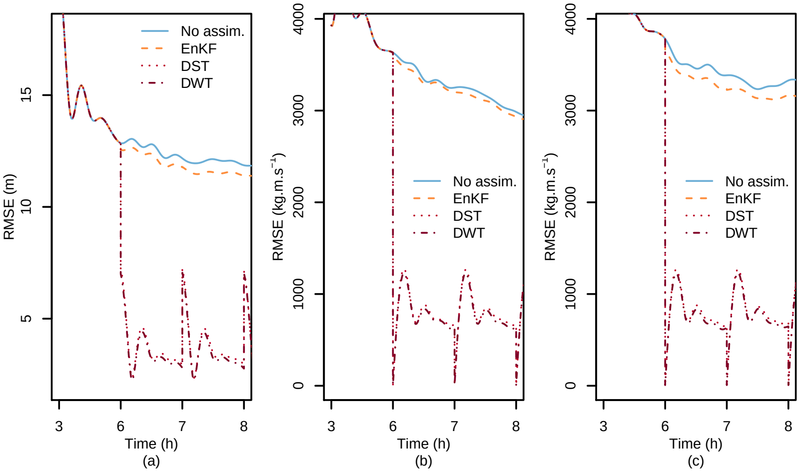
(a)



(b)

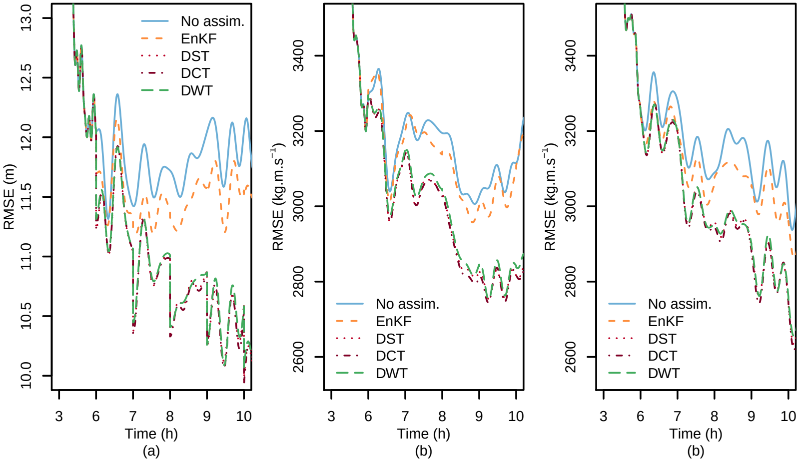
Shallow water equations

Whole state observed



Shallow water equations

Only height observed



Conclusions

- Data assimilation is usually computative/algorithmic challenging
- EnKF have problems with low rank approximation
- Our method can be used in every case, where the underlying hidden random field is second order stationary
- Research is still ongoing, wavelet transformation is under review

Questions, remarks...

Thank you!